CSE 598: Assignment 2

Joshua J. Daymude

Due Friday, October 11, 11:59pm

This is an individual assignment. You are welcome to discuss approaches with peers, but it should be clear that your writeup reflects your own understanding and effort. Your solutions will be evaluated based on *correctness* and *clarity*. Prepare your submission in LaTeX. Submit a `.zip` file on Canvas containing your final `.pdf` and `.tex` source file.

Problem 1

Consider the Markov chains depicted in Figure 1, and suppose each chain is initially in state A. For each chain, determine (i) if it is irreducible, aperiodic or both (ergodic), (ii) what its stationary distribution(s) is/are, if any, and (iii) if it is time-reversible with respect to its stationary distribution(s), if applicable. Explain your answers.

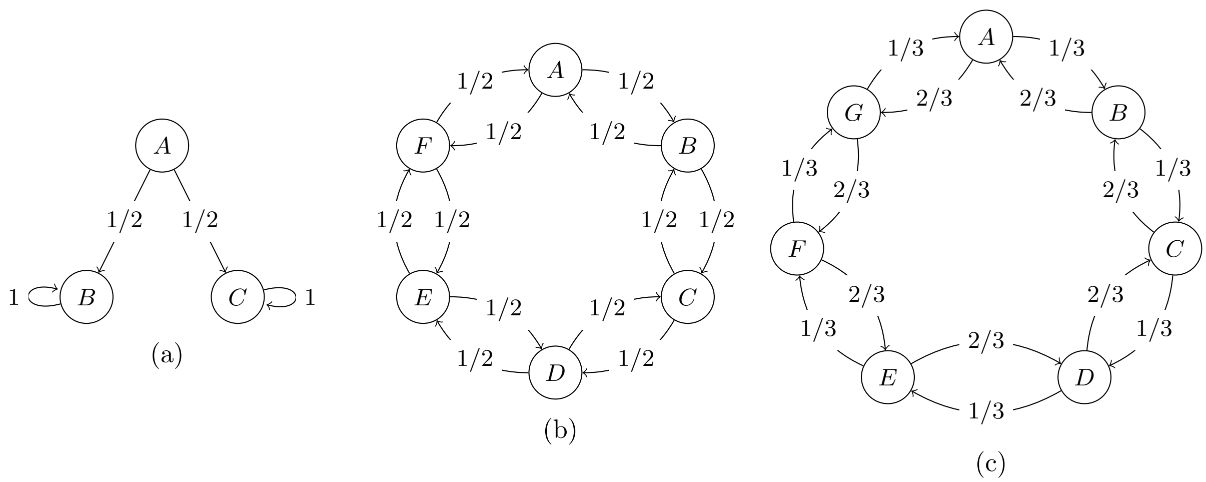

Figure 1: Markov chains for Problem 1.

Problem 2

Let G be a connected undirected graph. Prove that a random walk on G is periodic if and only if G is bipartite.

Problem 3

A transition matrix P is *doubly-stochastic* if every row and every column of P each sum to 1, i.e.:

$$\forall x \in \Omega, \ \sum_{y\in\Omega} P(x,y) = \sum_{y\in\Omega} P(y,x) = 1.$$

(a) Prove that if the transition matrix of a Markov chain is doubly-stochastic, then it has a uniform stationary distribution.

(b) Use (a) to prove that the Markov chain for top-to-random shuffling (Algorithm 6) has a uniform stationary distribution.

(c) Does the converse hold? That is, if a Markov chain has a uniform stationary distribution then does it have a doubly-stochastic transition matrix? Either prove this to be true or provide a counterexample.

Problem 4

Recall that the Fundamental Theorem of Markov Chains (Theorem 3.6) states that if a Markov chain $\mathcal{M} = (\Omega, P)$ is finite and ergodic (i.e., irreducible and aperiodic), then it has a unique stationary distribution π satisfying:

$$\forall x, y \in \Omega, \ \lim_{t\to\infty} P^t(x,y) = \pi(y).$$

Here, we prove the existence of such a distribution π.

(a) For distinct states $x, y \in \Omega$, let $v_x(y)$ be the expected number of times a chain, starting from x, visits y before its first return to x. Assume $v_x(x) = 1$. Prove that $\forall x \in \Omega$, the elements of vector $v_x(\cdot)$ are finite and strictly positive. [Hint: Use the definition of ergodicity to obtain a time t such that $P^t(x,y) > 0$ for all x, y.]

(b) Prove that $\forall x \in \Omega$, $v_x(\cdot)$ satisfies $v_x(\cdot)P = v_x(\cdot)$. [Hint: It may be useful to expand the expectation as $v_x(y) = \sum_{t\geq 1} v_x^{(t)}(y)$, where $v_x^{(t)}$ is the probability that the chain is at y exactly t steps after leaving x.]

(c) Give a (very) brief explanation of how (a) and (b) prove the existence of a unique stationary distribution π.

Problem 5

The *East model* is a Markov chain on state space $\Omega = \{x = x_1x_2\cdots x_{n+1} \in \{0,1\}^{n+1} : x_{n+1} = 1\}$ with the following transitions:

1. Select $i \in \{1, 2, \ldots, n\}$ uniformly at random.
2. If $x_{i+1} = 1$, then set $x_i \leftarrow 1 - x_i$ (i.e., flip the i-th bit); otherwise, do nothing.

Prove the following:

(a) This chain is ergodic.

(b) The unique stationary distribution of this chain is uniform.

(c) The mixing time of this chain is at least $n^2 - cn^{3/2}$ for some constant c. [Hint: To show $T_{mix}(\varepsilon) \geq T$, you need to show that after T iterations, the total variation distance from the stationary distribution is still more than ε. Starting from any configuration, how long does it take before the leftmost 1 has moved once to the left? Starting from $0^n 1$, how long does it take for the leftmost bit to be 1? Use Chebyshev's inequality and the fact that a geometric random variable with parameter p has expectation $1/p$ and variance $1/p^2 - 1/p$.]

Problem 6

Let $n, k \in \mathbb{Z}^+$ with $k \leq n/2$, and let Ω be the set of all subsets of $\{1, 2, \ldots, n\}$ of cardinality k. Then $|\Omega| = \binom{n}{k}$. Consider a Markov chain on Ω with the following transitions: from subset $S_t \in \Omega$, repeat:

1. Choose an element $a \in S_t$ and an element $b \in \{1, 2, \ldots, n\} \setminus S_t$ each uniformly at random.
2. Set $S_{t+1} \leftarrow S \setminus \{a\} \cup \{b\}$.

Prove the following:

(a) This chain is ergodic. [Hint: If this chain needs to be augmented with self-loop probabilities to achieve aperiodicity, state how you would do this.]

(b) The unique stationary distribution of this chain is uniform.

(c) The mixing time of this chain is $\mathcal{O}(n \log k)$. [Hint: Use a coupling argument.]