CSE 598: Assignment 3

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DUE WEDNESDAY, NOVEMBER 27, START OF CLASS

This is an individual assignment. You are welcome to discuss approaches with peers, but it should be clear that your writeup reflects your own understanding and effort. Your solutions will be evaluated based on *correctness* and *clarity*. Prepare your submission in LaTeX. Submit a `.pdf` file on Canvas. Unless otherwise specified, any bounds on T_{mix} are for $T_{mix}(1/4)$.

Problem 1

Recall the barbell graph $G = (V, E)$ composed of two components, $A = K_{n/3}$ and $B = K_{2n/3}$ connected by a single edge. Using conductance, we showed that the lazy random walk on this graph has $T_{mix} \geq n^2/18$. Use a coupling argument to prove an asymptotically matching upper bound on T_{mix}, showing that $T_{mix} = \Theta(n^2)$.

[Hint: Compose two couplings. If X_t and Y_t are in different components, lower bound the probability that a naive coupling (i.e., where X_t and Y_t update independently) brings them into the same component in a constant number of steps. Then lower bound the probability that an identity coupling (where the same vertex is chosen for both chains to move to) couples the chains in another constant number of steps, as in Section 6.1.1. Use the conjunction of these two probabilities to show the probability the two chains have not coupled after some longer number of steps is small.]

Problem 2

Recall that the Ising model considers an $n \times n$ square grid $G = (V, E)$ and the states are $\Omega = \{-1, +1\}^V$, i.e., all the ways of assigning + or − spins to the vertices of G. We saw how to construct transitions based on the Metropolis process; here, we consider another Markov chain that samples from the same distribution. Let β be inverse temperature, and let $\lambda = e^{2\beta}$ for convenience. Then the transitions from a state $X_t \in \Omega$ are as follows:

1. Choose a vertex $u \in V$ uniformly at random.
2. Let $d^+(u)$ be the number of neighbors v of u with $X_t(v) = +1$; define $d^-(u)$ analogously. With probability $\lambda^{d^+(u)}/(\lambda^{d^+(u)} + \lambda^{d^-(u)})$, set $X_{t+1}(u) \leftarrow +1$; otherwise, set $X_{t+1}(u) \leftarrow -1$.
3. For all $v \neq u$, set $X_{t+1}(v) \leftarrow X_t(v)$ and $Y_{t+1}(v) \leftarrow Y_t(v)$.

Use path coupling to prove that for sufficiently small λ (or equivalently, sufficiently small β or sufficiently high temperature), this Markov chain has $T_{mix} = \mathcal{O}(n^2 \log n)$.

Problem 3

Consider the lazy random walk on complete binary tree $T = (V, E)$ on n vertices with root vertex v_0 and depth k (i.e., $n = 2^{k+1} - 1$). Prove the following results to conclude that $T_{mix} \geq (n-2)/2$.

(a) Verify the more general result that $\pi(v) = d(v)/2|E|$, where $d(v)$ is the degree of vertex v, is the stationary distribution of both the non-lazy and lazy random walks on a graph $G = (V, E)$.

(b) Let S consist of the right child of root v_0 and all of its descendants. Prove that $\pi(S) < 1/2$.

(c) Use a conductance argument to prove that $T_{mix} \geq (n-2)/2$.

Problem 4

Consider the random walk on the complete graph K_n. The transitions are:

$$P(x, y) = \begin{cases} 1/(n-1) & \text{if } x \neq y; \\ 0 & \text{otherwise.} \end{cases}$$

(a) For $i \in \{2, \ldots, n\}$, define a vector v_i as follows:

$$v_i(j) = \begin{cases} -1 & \text{if } j = 1; \\ 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $v_2, \ldots, v_n$ are the nontrivial eigenvectors of P and that the corresponding eigenvalues are $\lambda_2 = \cdots = \lambda_n = -1/(n-1)$.

(b) Consider the lazy version of this random walk with transition matrix Q. What are the eigenvalues of Q? What is its spectral gap? Prove an upper and lower bound on T_{mix}.

Problem 5

Consider shuffling a deck of n cards by transposition. The state space Ω consists of all $n!$ permutations of n cards; let $x \in \Omega$ be written as $x = (x_1, x_2, \ldots, x_n)$ where x_i is the card at the i-th position in x. Define the transitions as follows:

1. Select $i, j \in \{1, 2, \ldots, n\}$ uniformly at random (without replacement).
2. With probability $1/2$, swap x_i and x_j; otherwise, do nothing.

For each pair of distinct states $x, y \in \Omega$, define the canonical path γ_{xy} as follows. For each $i \in \{1, 2, \ldots, n\}$, move y_i from its current position to position i (if it is not there already) by switching it with the card currently in position i.

(a) Define an injective (one-to-one) mapping from canonical paths γ_{xy} that use a particular (non-self-loop) transition t to Ω, thereby showing that the number of paths that use any particular transition is at most $|\Omega|$.

(b) Using (a), prove that $T_{mix}(\varepsilon) = \mathcal{O}(n^3(\log \varepsilon^{-1} + n \log n))$.