A Local Stochastic Algorithm for Separation in Heterogeneous Self-Organizing Particle Systems

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Programmable Active Matter

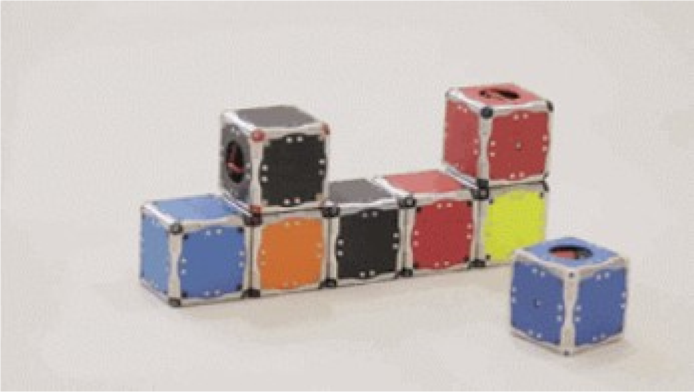
Programmable matter is a substance that can change its physical properties autonomously based on user input or environmental stimuli.

Composed of **active particles** that can control their decisions and movements.

“Catoms”

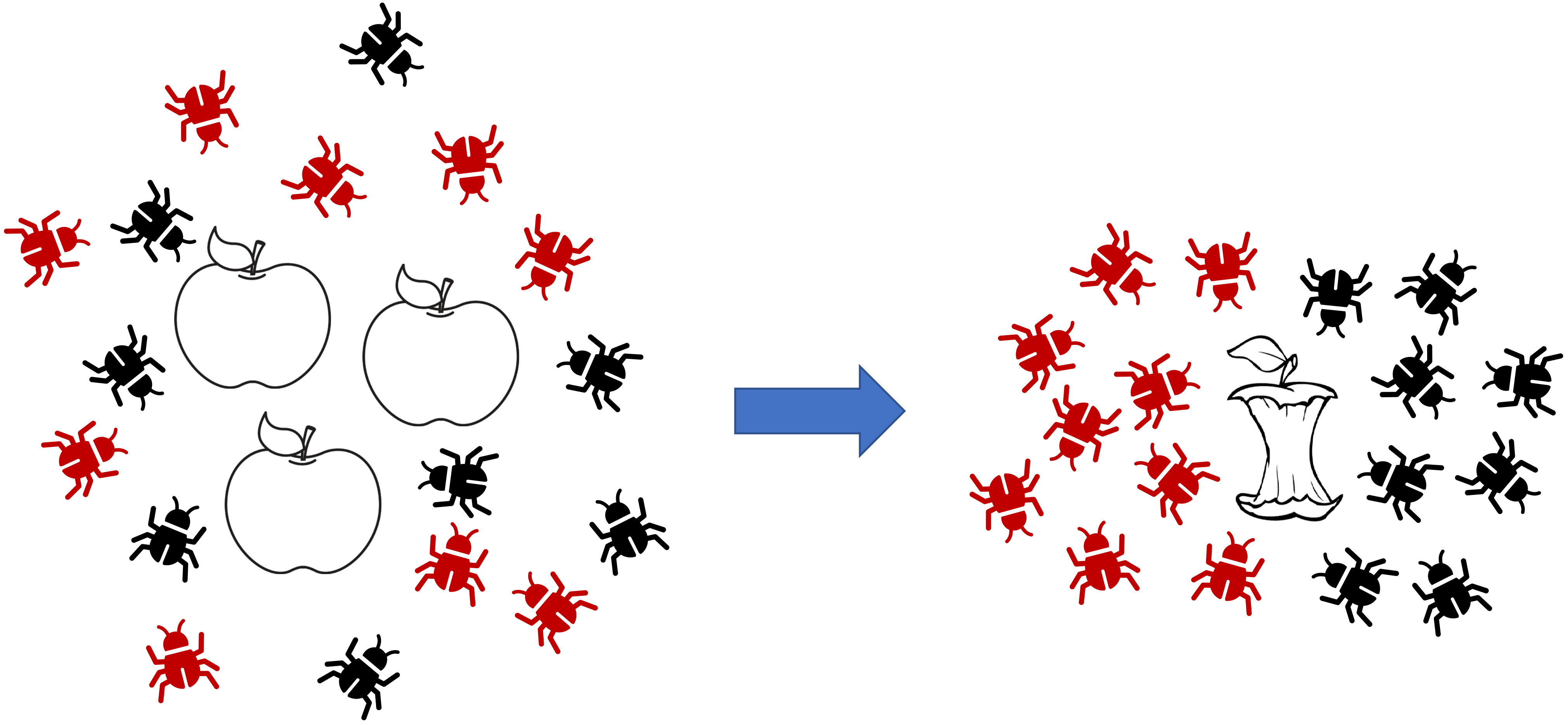
“Kilobots”

“M-Blocks”



“Particle Robots”

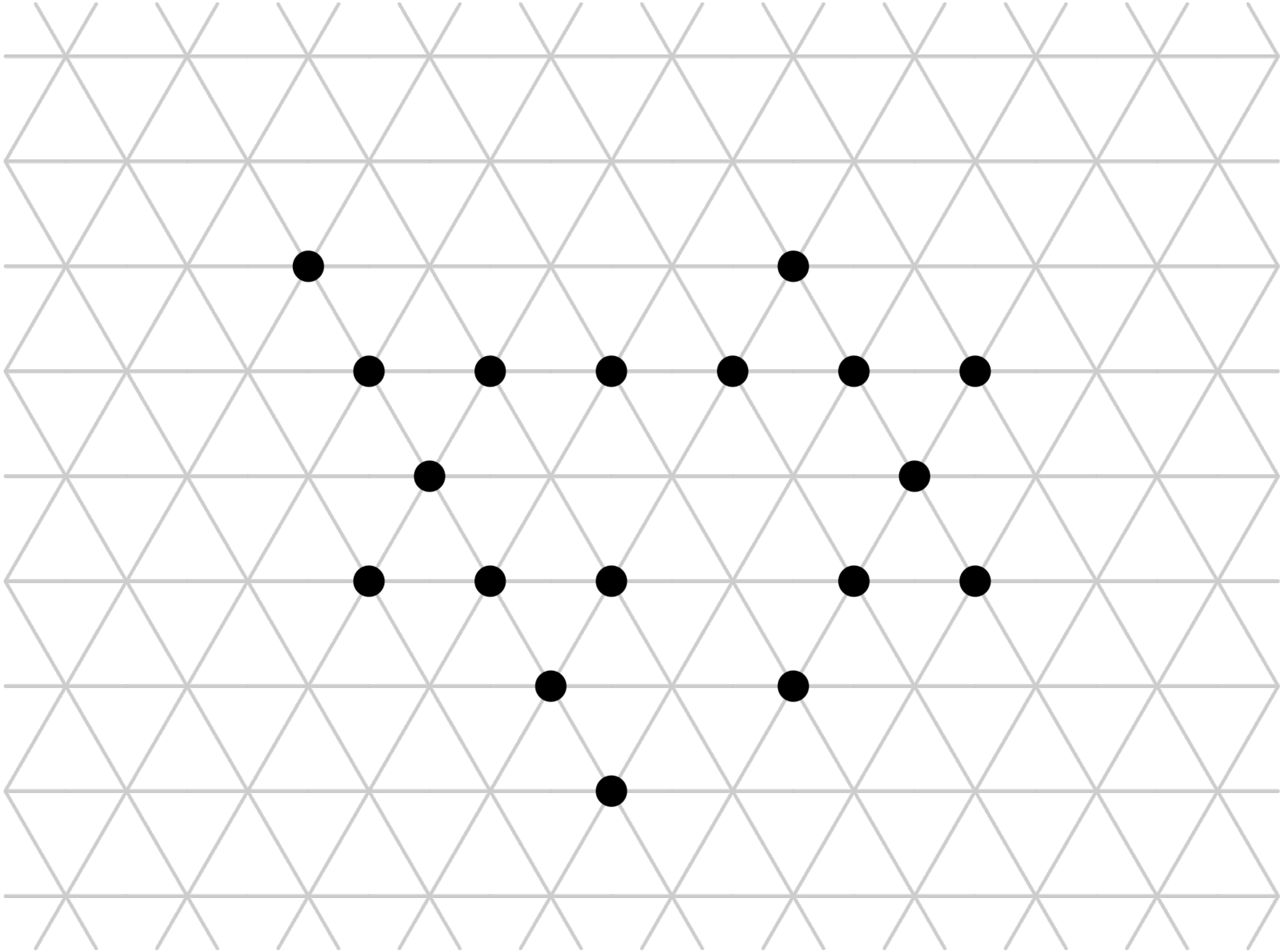
Separation in Biology



"Integrated" "Separated"

Self-Organizing Particle Systems

Abstracts programmable matter as simple computational "particles" that use **distributed, local algorithms** to achieve **system-level goals**.



The Geometric Amoebot Model

- Particles occupy nodes of the triangular lattice and move along edges.
- Local communication, only with immediate neighbors.
- Constant-size memory per particle.
- No global information (coordinates, orientation, etc.)

Algorithms in the Amoebot Model

Each particle independently and concurrently runs its own instance of the given distributed algorithm to achieve system-level goals:

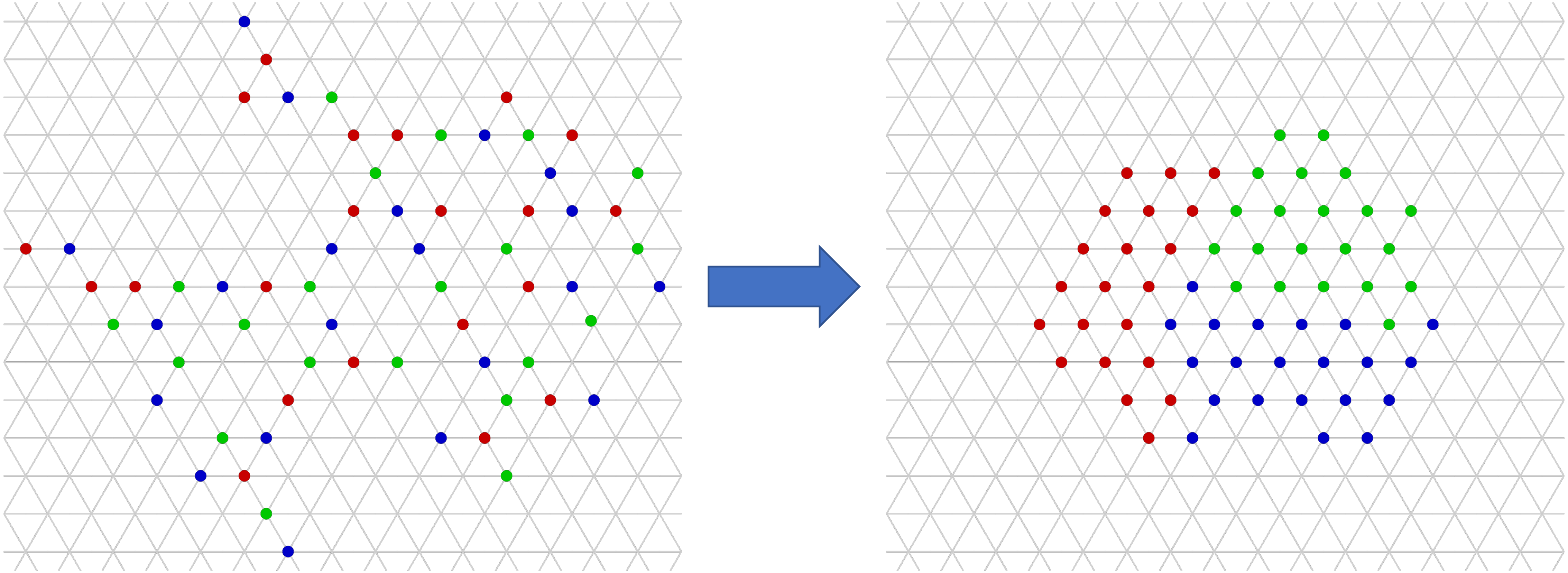
- **Shape Formation** [Derakhshandeh, Gmyr, Richa, Scheideler, Strothmann 2015-16]
- **Object Coating** [D., Derakhshandeh, Gmyr, Porter, Richa, Scheideler, Strothmann 2017-18]
- **Leader Election** [D., Gmyr, Richa, Scheideler, Strothmann 2017]
- **Compression & Expansion** [Cannon, D., Randall, Richa 2016]
- **Separation & Integration** [Cannon, D., Gökmen, Randall, Richa 2019]

Our Goal

Compression: Gather the particle system together.

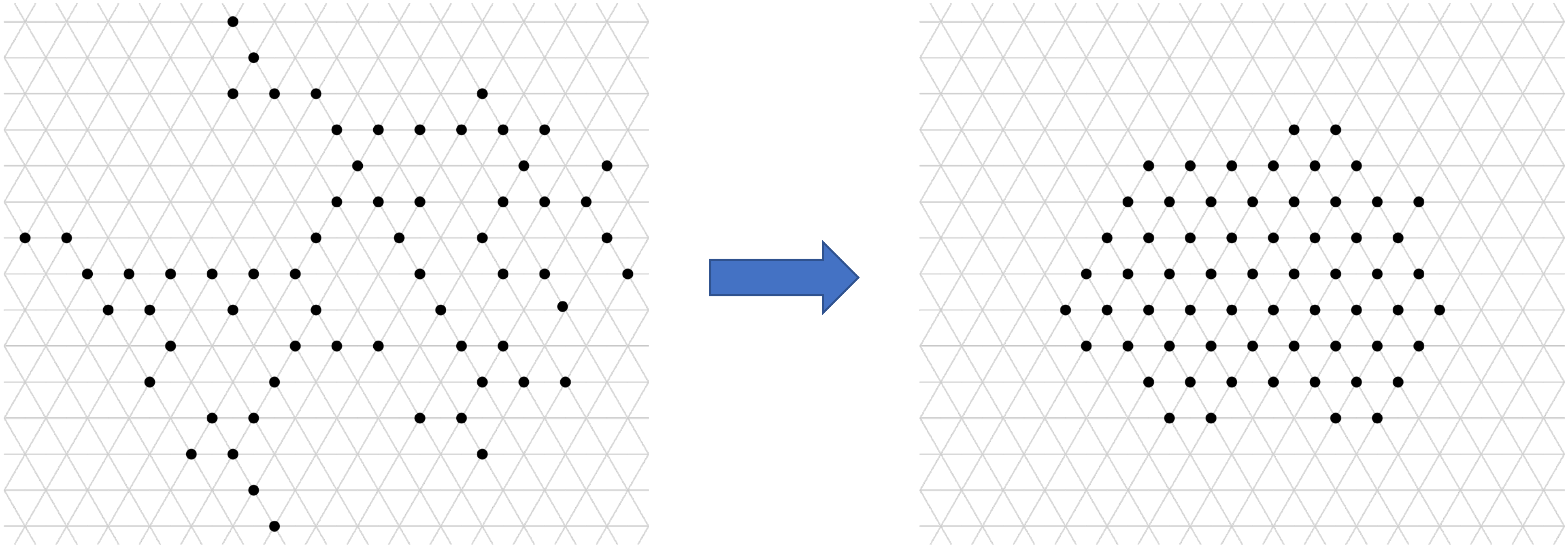
Compression + Separation: Gather together overall and by color.

- For our analysis, we consider the 2-color case.



Compression

Question: Using local, distributed rules, how can particles “compress,” gathering together?



Not compressed Compressed

Definition: A configuration is **α-compressed** if its perimeter is at most **α** times the minimum perimeter (for this number of particles).

Compression: Algorithm

[Cannon, D., Randall, Richa 2016]

This distributed, stochastic algorithm for compression:

- Ensures system connectivity on the triangular lattice.
- Uses Poisson clocks to activate particles (no synchronization).
- Uses Metropolis probabilities to converge to $\pi(\sigma) \propto \lambda^{e(\sigma)}$, for bias parameter $\lambda > 1$.

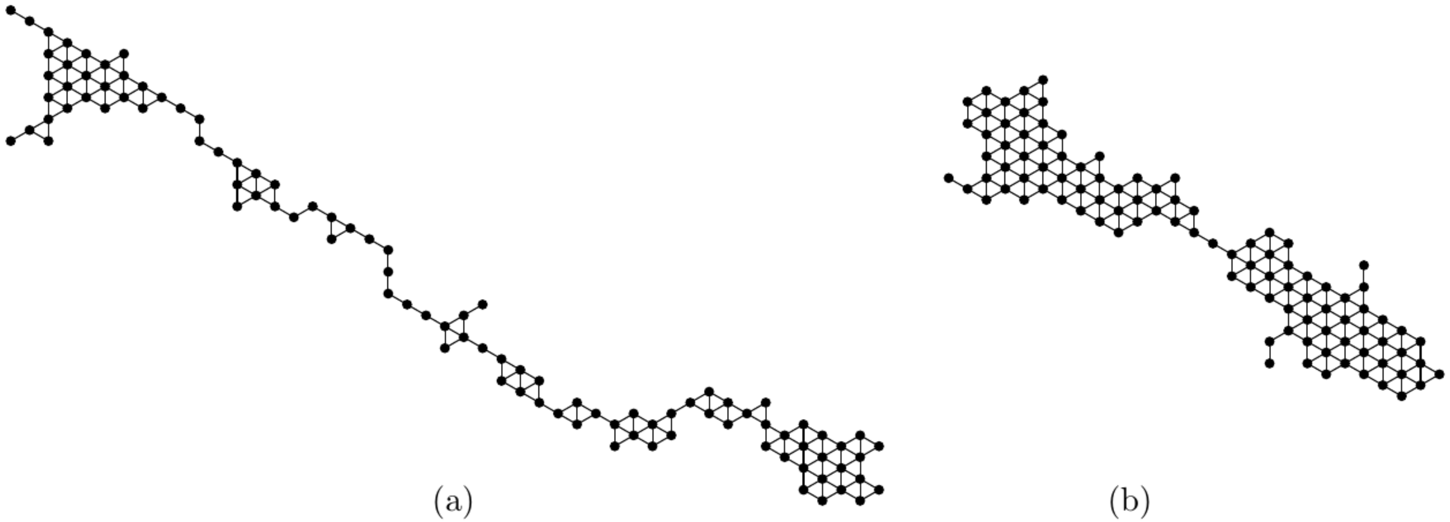
Fix $\lambda > 1$. Start in any connected configuration.

When a particle activates (according to its Poisson clock), do:

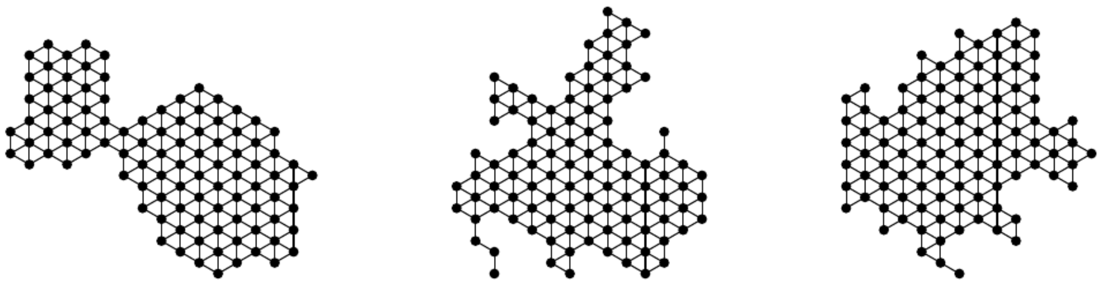
1. Pick a random neighboring node.
2. If the proposed node is unoccupied, move with probability $\min\{\lambda^{\Delta e}, 1\}$.
3. Otherwise, do nothing.

Compression: Simulations, $\lambda = 4$

[Cannon, D., Randall, Richa 2016]



(a) (b)

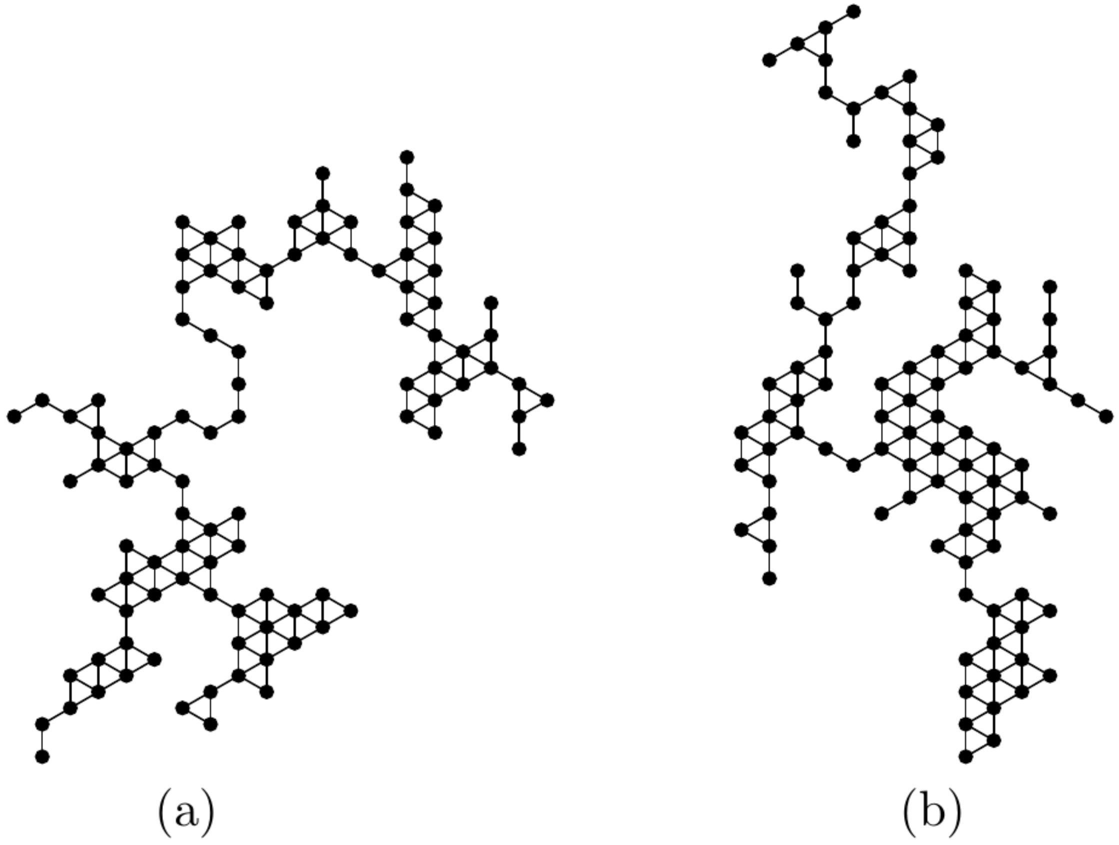


(c) (d) (e)

100 particles after (a) 1 million, (b) 2 million, (c) 3 million, (d) 4 million, and (e) 5 million iterations.

Compression: Simulations, $\lambda = 2$

[Cannon, D., Randall, Richa 2016]



100 particles after (a) 10 million and (b) 20 million iterations.

Compression: Results

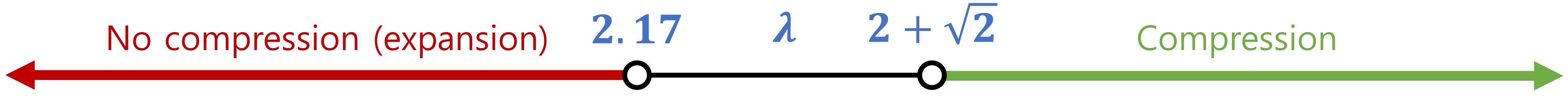
[Cannon, D., Randall, Richa 2016]

Definition: A configuration is **α-compressed** if its perimeter is at most **α** times the minimum perimeter (for this number of particles).

Theorem: When $\boldsymbol{\lambda} > \mathbf{2} + \sqrt{\mathbf{2}}$, there exists an $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\lambda})$ such that the particle system is α-compressed at stationarity almost surely.

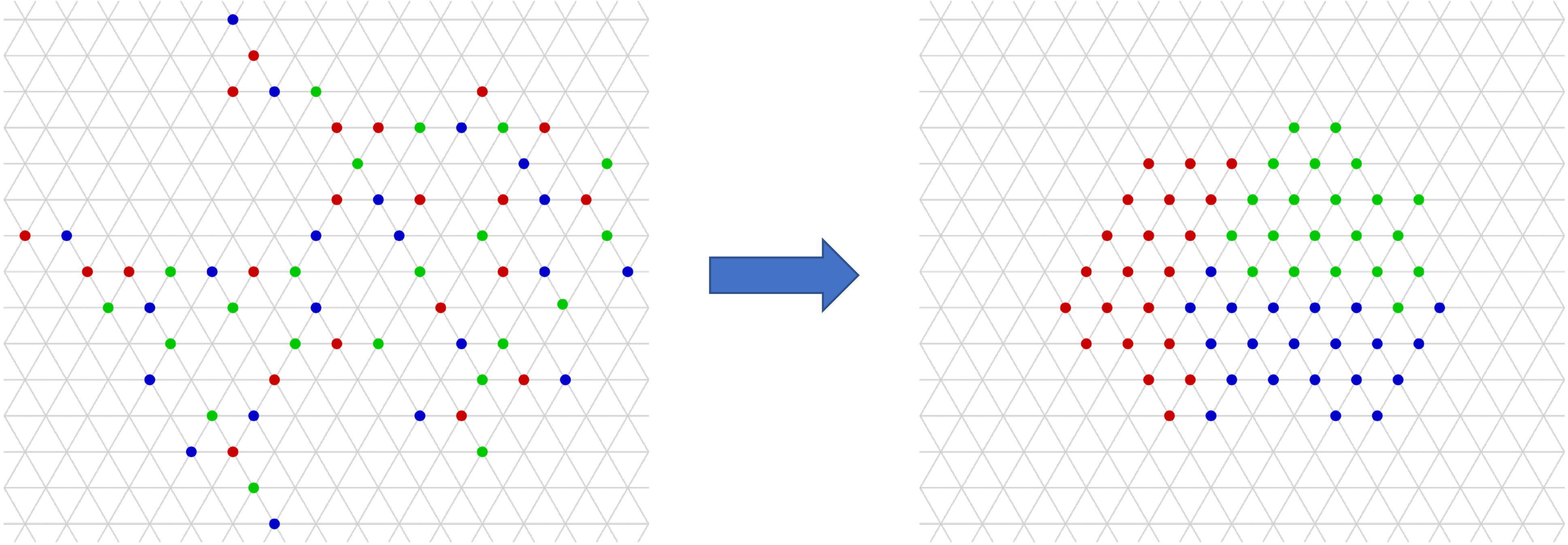
- E.g., when $\lambda = 4$, we have $\alpha = 9$.

Theorem: When $\boldsymbol{\lambda} < \mathbf{2.17}$, for any $\boldsymbol{\alpha} > \mathbf{1}$, the probability the particle system is α-compressed at stationarity is exponentially small.



Separation

Question: Using local, distributed rules, how can heterogeneous particles "compress" overall while also separating into mostly monochromatic groups?



Neither compressed nor separated

Compressed and separated

Separation: Algorithm

This distributed, stochastic algorithm for separation:

- Like compression, ensures system connectivity and is not synchronized.
- Uses Metropolis probabilities to converge to $\pi(\sigma) \propto \lambda^{e(\sigma)} \cdot \gamma^{m(\sigma)}$, for bias parameters λ, γ.

Fix λ and γ. Start in any connected configuration.

When a particle activates (according to its Poisson clock), do:

1. Pick a random neighboring node.
2. Move with probability $\min\{\lambda^{\Delta e} \cdot \gamma^{\Delta m}, 1\}$.
3. Otherwise, do nothing.

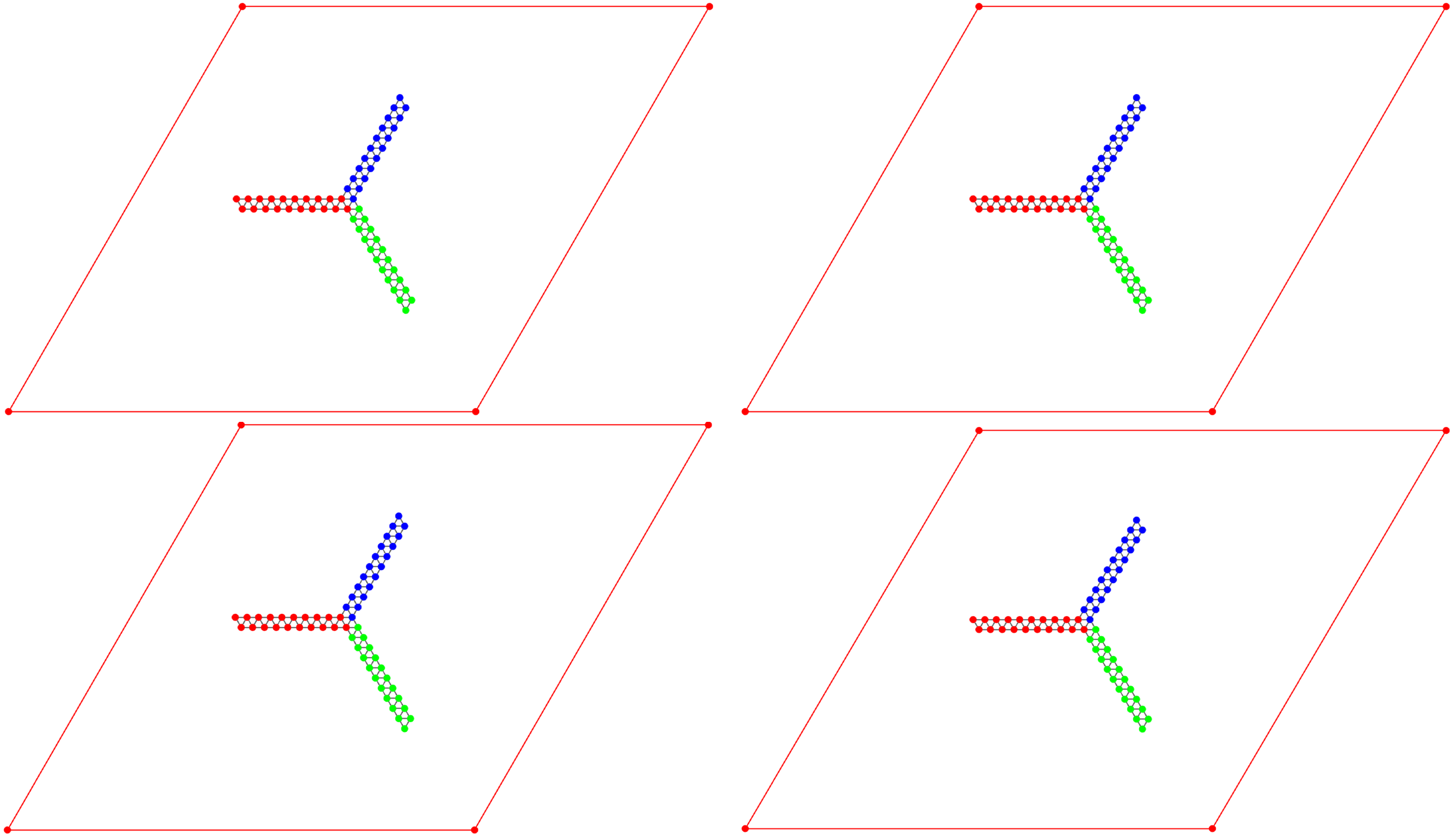
Separation: Simulations

Integration: $\gamma = 0.25$

Separation: $\gamma = 4$

Compression: $\lambda = 4$

Expansion: $\lambda = 1$



Results: Separation for large γ

Stationary distribution $\pi(\sigma) \propto \lambda^{e(\sigma)} \cdot \gamma^{m(\sigma)} = (\lambda\gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}$.

Theorem: When $\boldsymbol{\lambda\gamma > 6.83}$ and $\boldsymbol{\gamma > 5.66}$, there exists an $\boldsymbol{\alpha = \alpha(\lambda, \gamma)}$ such that the particle system is α-compressed at stationarity almost surely.

Proof techniques. Uses the cluster expansion and a Peierls argument.

Theorem: Moreover, separation occurs among the α-compressed configurations at stationarity almost surely.

Proof techniques. Uses bridging [Miracle, Pascoe, Randall 2011] and a Peierls argument.

Results: Separation for large γ

Theorem: When $\boldsymbol{\lambda\gamma > 6.83}$ and $\boldsymbol{\gamma > 5.66}$, there exists an $\boldsymbol{\alpha = \alpha(\lambda, \gamma)}$ such that the particle system is α-compressed at stationarity almost surely.

<u>Proof sketch.</u>

Stationary distribution $\pi(\sigma) = (\lambda\gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}/Z$.

Let S_α be the non-α-compressed configurations. Want to show $\pi(S_\alpha)$ is exponentially small.

Partition S_α into sets of configurations A_k with the same perimeter k. Then:

$$\pi(A_k) = \sum_{\sigma \in A_k} (\lambda\gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}/Z$$

$$= (\lambda\gamma)^{-k} \cdot \sum_{\sigma \in A_k} \gamma^{-h(\sigma)}/Z$$

Results: Separation for large γ

Theorem: When $\boldsymbol{\lambda\gamma > 6.83}$ and $\boldsymbol{\gamma > 5.66}$, there exists an $\boldsymbol{\alpha = \alpha(\lambda, \gamma)}$ such that the particle system is α-compressed at stationarity almost surely.

Proof sketch (cont.)

So $\pi(A_k) = (\lambda\gamma)^{-k} \cdot \sum_{\sigma \in A_k} \gamma^{-h(\sigma)}/Z$.

However, while true in the uncolored case (compression), this is not true for our heterogeneous setting.

If we had $\sum_{\sigma \in A_k} \gamma^{-h(\sigma)} \leq b^k$ for some $b > 1$, then:

$$\pi(S_\alpha) = \sum_{k=\alpha \cdot p_{min}}^{p_{max}} \pi(A_k) = \sum_{k=\alpha \cdot p_{min}}^{p_{max}} (\lambda\gamma)^{-k} \cdot \sum_{\sigma \in A_k} \gamma^{-h(\sigma)}/Z \leq \sum_{k=\alpha \cdot p_{min}}^{p_{max}} (\lambda\gamma)^{-k} \cdot b^k/Z$$

Lemma [Volume-Surface Decomposition]: When $\gamma > 5.66$, there are a and b such that:

$$a^n \cdot b^{-k} \leq \sum_{\sigma \in A_k} \gamma^{-h(\sigma)} \leq a^n \cdot b^k$$

Results: Separation for large γ

Lemma [Volume-Surface Decomposition]: If Ω_Λ are all 2-colorings with monochrome perimeter of an uncolored configuration Λ and $\gamma > 5.66$, then there are a and b such that:

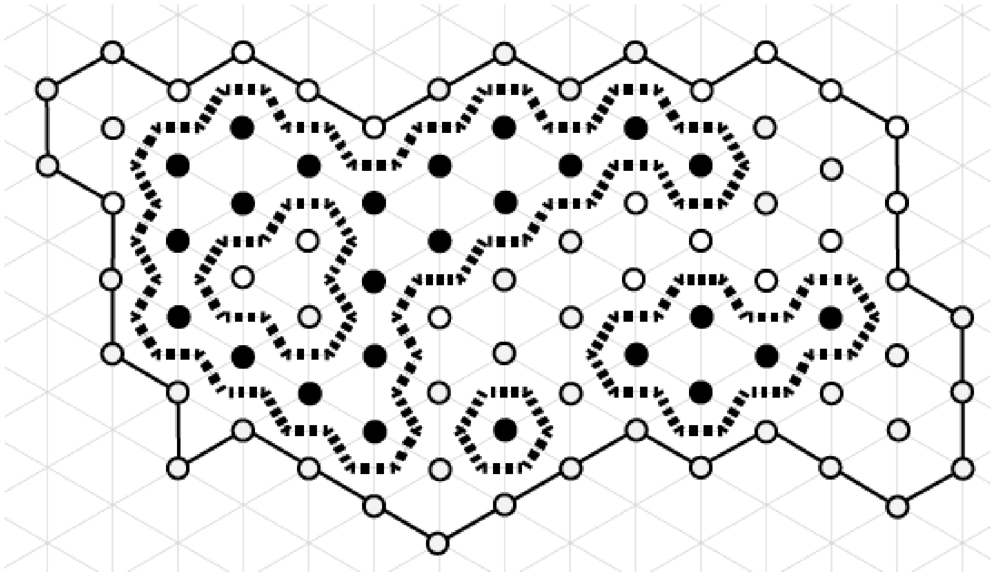
$$a^n \cdot b^{-k} \leq \sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)} \leq a^n \cdot b^k$$

Proof sketch (cont.)

Express $\sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)}$ as a "polymer model."

- An interface I between two color classes is a loop.
- Let Γ_Λ be the set of all interfaces in Λ. Then:

$$\sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)} = \sum_{\text{pairwise disjoint } \Gamma' \in \Gamma_\Lambda} \prod_{I \in \Gamma'} \gamma^{-|I|}$$



Results: Separation for large γ

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<u>Proof sketch (cont.)</u>

$$\sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)} = \sum_{\text{pairwise disjoint } \Gamma' \in \Gamma_\Lambda} \prod_{I \in \Gamma'} \gamma^{-|I|}$$

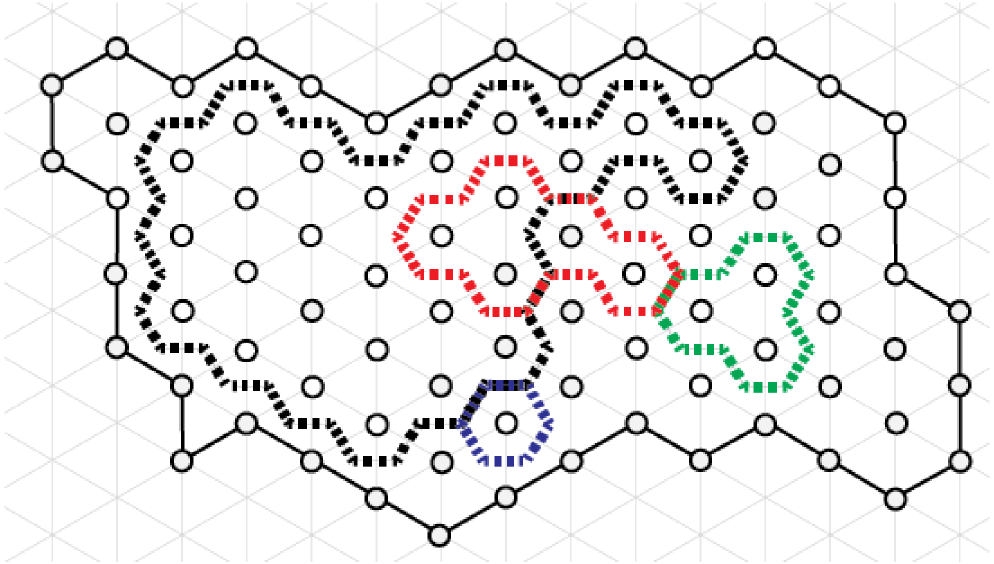
A cluster is a multiset $X \subseteq \Gamma_\Lambda$ of connected interfaces.

The cluster expansion for our quantity is:

$$\ln\left(\sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)}\right) = \sum_{X \subseteq \Gamma_\Lambda} \phi(X) \prod_{I \in X} \gamma^{-|I|}$$

Need to know this formal series converges.

Also need to bound this sum.



Results: Separation for large γ

Lemma [Volume-Surface Decomposition]: If Ω_Λ are all 2-colorings with monochrome perimeter of an uncolored configuration Λ and $\gamma > 5.66$, then there are a and b such that:

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Proof sketch (cont.)

$$\sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)} = \sum_{\text{pairwise disjoint } \Gamma' \in \Gamma_\Lambda} \prod_{I \in \Gamma'} \gamma^{-|I|}$$

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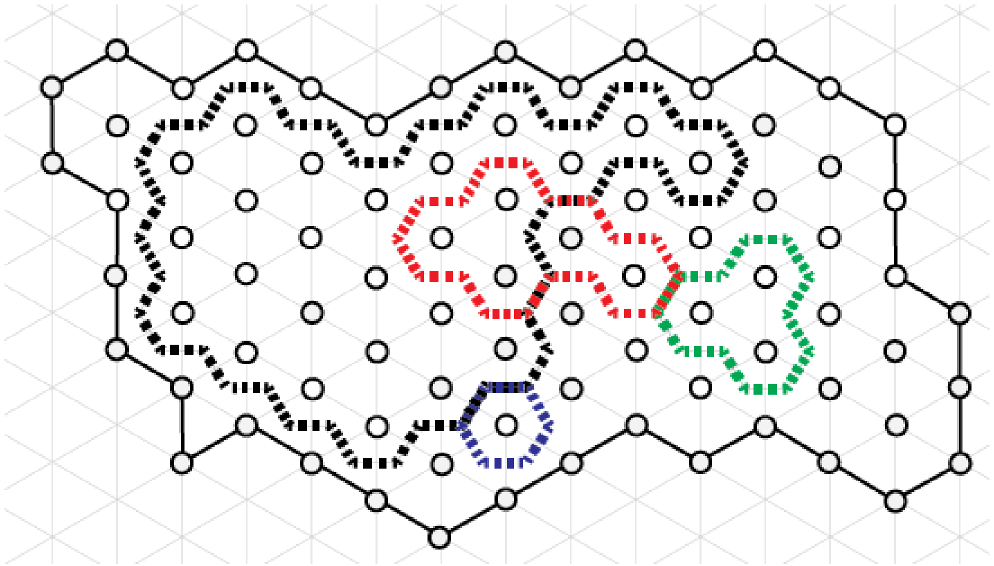
Using the Kotecký-Preiss condition with $\gamma > 5.66$ and a constant $c = 0.0001$, we show series convergence and:

$$a^n \cdot e^{-ck} \leq \sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)} \leq a^n \cdot e^{ck}$$



Results: Integration for γ close to 1

Stationary distribution $\pi(\sigma) \propto \lambda^{e(\sigma)} \cdot \gamma^{m(\sigma)} = (\lambda\gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}$.

Theorem: When $\boldsymbol{\lambda(\gamma+1) > 6.83}$ and $\mathbf{0.98 \leq \boldsymbol{\gamma} \leq 1.02}$, there exists an $\boldsymbol{\alpha = \alpha(\lambda, \gamma)}$ such that the particle system is α-compressed at stationarity almost surely.

Theorem: Moreover, separation occurs among the α-compressed configurations at stationarity with exponentially small probability.

Results: Integration for γ close to 1

Theorem: When $\boldsymbol{\lambda(\gamma+1)>6.83}$ and $\boldsymbol{0.98\le\gamma\le 1.02}$, there exists an $\boldsymbol{\alpha=\alpha(\lambda,\gamma)}$ such that the particle system is α-compressed at stationarity almost surely.

<u>Proof sketch.</u>

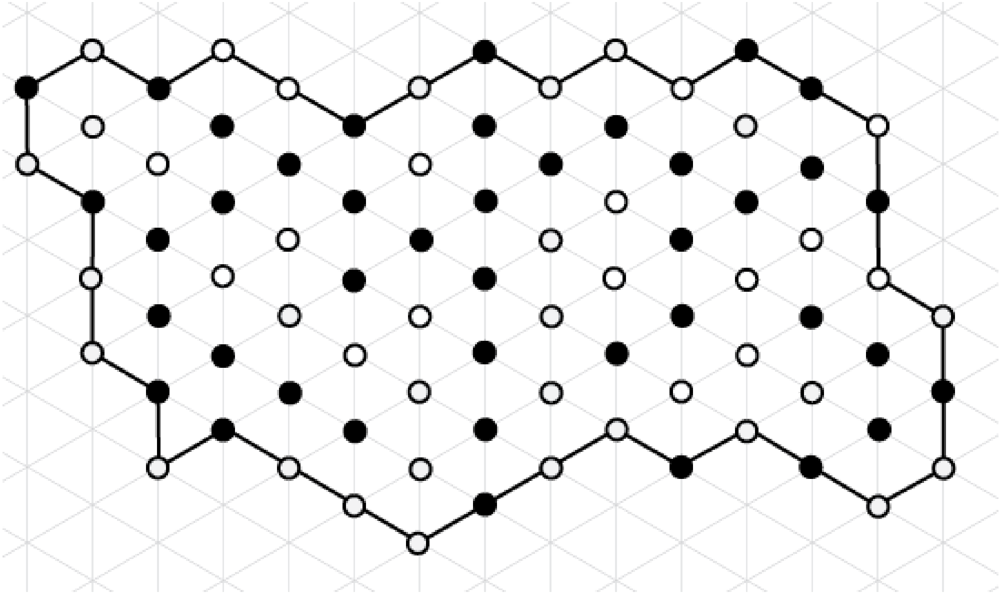
Recall: $\sum_{\sigma\in\Omega_\Lambda}\gamma^{-h(\sigma)}=\sum_{\text{pairwise disjoint }\Gamma'\in\Gamma_\Lambda}\prod_{I\in\Gamma'}\gamma^{-|I|}$

The $\gamma^{-|I|}$ term does not decay fast enough when γ is close to 1.

Rewrite using the high temperature expansion.

$$\sum_{\sigma\in\Omega_\Lambda}\gamma^{-h(\sigma)}=(\dots)\sum_{\text{even }E\subseteq E(\Lambda)}\left(\frac{\gamma-1}{\gamma+1}\right)^{|E|}$$

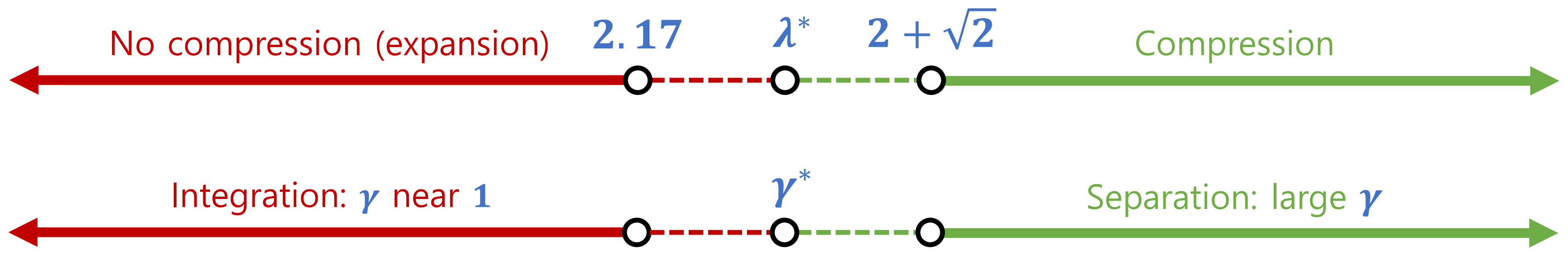
Then apply the cluster expansion and Peierls argument similar to the previous proof.



Open Questions

1. What is the mixing time of our algorithms?
   - Connections to the low temperature plus-boundary Ising model on $\mathbb{Z}^2$ suggests proofs are hard.
   - However, we observe compression in simulation after only $O(n^{3.3})$ iterations.

2. Are there critical values $\boldsymbol{\lambda}^*$ and $\boldsymbol{\gamma}^*$ marking phase transitions?



3. What other new ways can we use the cluster expansion?
   - Used to show aggregation/dispersion in the disconnected case. [Dutta, Li, Cannon, D., Aydin, Richa, Goldman, Randall]

Thank you!

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