A Local Stochastic Algorithm for Separation in Heterogeneous Self-Organizing Particle Systems

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Programmable Active Matter

Programmable matter is a substance that can change its physical properties autonomously based on user input or environmental stimuli.

Composed of active particles that can control their decisions and movements.
Separation in Biology

“Integrated”

“Separated”
Abstracts programmable matter as simple computational “particles” that use distributed, local algorithms to achieve system-level goals.

The Geometric Amoebot Model

- Particles occupy nodes of the triangular lattice and move along edges.
- Local communication, only with immediate neighbors.
- Constant-size memory per particle.
- No global information (coordinates, orientation, etc.)
Algorithms in the Amoebot Model

Each particle independently and concurrently runs its own instance of the given distributed algorithm to achieve system-level goals:

- **Shape Formation** [Derakhshandeh, Gmyr, Richa, Scheideler, Strothmann 2015-16]
- **Object Coating** [D., Derakhshandeh, Gmyr, Porter, Richa, Scheideler, Strothmann 2017-18]
- **Leader Election** [D., Gmyr, Richa, Scheideler, Strothmann 2017]
- **Compression & Expansion** [Cannon, D., Randall, Richa 2016]
- **Separation & Integration** [Cannon, D., Gökmen, Randall, Richa 2019]
Our Goal

**Compression:** Gather the particle system together.

**Compression + Separation:** Gather together overall and by color.

- For our analysis, we consider the 2-color case.
Compression

**Question:** Using local, distributed rules, how can particles “compress,” gathering together?

**Definition:** A configuration is \( \alpha \)-compressed if its perimeter is at most \( \alpha \) times the minimum perimeter (for this number of particles).
Compression: Algorithm

[Cannon, D., Randall, Richa 2016]
This distributed, stochastic algorithm for compression:

• Ensures system connectivity on the triangular lattice.
• Uses Poisson clocks to activate particles (no synchronization).
• Uses Metropolis probabilities to converge to $\pi(\sigma) \propto \lambda^e(\sigma)$, for bias parameter $\lambda > 1$.

Fix $\lambda > 1$. Start in any connected configuration.

When a particle activates (according to its Poisson clock), do:

1. Pick a random neighboring node.
2. If the proposed node is unoccupied, move with probability $\min\{\lambda^e, 1\}$.
3. Otherwise, do nothing.
Compression: Simulations, $\lambda = 4$

[Cannon, D., Randall, Richa 2016]

100 particles after (a) 1 million, (b) 2 million, (c) 3 million, (d) 4 million, and (e) 5 million iterations.
Compression: Simulations, $\lambda = 2$

[Cannon, D., Randall, Richa 2016]

100 particles after (a) 10 million and (b) 20 million iterations.
**Compression: Results**

[Cannon, D., Randall, Richa 2016]

**Definition**: A configuration is $\alpha$-compressed if its perimeter is at most $\alpha$ times the minimum perimeter (for this number of particles).

**Theorem**: When $\lambda > 2 + \sqrt{2}$, there exists an $\alpha = \alpha(\lambda)$ such that the particle system is $\alpha$-compressed at stationarity almost surely.

- E.g., when $\lambda = 4$, we have $\alpha = 9$.

**Theorem**: When $\lambda < 2.17$, for any $\alpha > 1$, the probability the particle system is $\alpha$-compressed at stationarity is exponentially small.
**Question:** Using local, distributed rules, how can heterogeneous particles “compress” overall while also separating into mostly monochromatic groups?
This distributed, stochastic algorithm for separation:

• Like compression, ensures system connectivity and is not synchronized.
• Uses Metropolis probabilities to converge to \( \pi(\sigma) \propto \lambda^e(\sigma) \cdot \gamma^m(\sigma) \), for bias parameters \( \lambda, \gamma \).

Fix \( \lambda \) and \( \gamma \). Start in any connected configuration.

When a particle activates (according to its Poisson clock), do:

1. Pick a random neighboring node.
2. Move with probability \( \min\{\lambda^e \cdot \gamma^m, 1\} \).
3. Otherwise, do nothing.
Separation: Simulations

Integration: $\gamma = 0.25$

Separation: $\gamma = 4$

Compression: $\lambda = 4$

Expansion: $\lambda = 1$
Results: Separation for large $\gamma$

Stationary distribution $\pi(\sigma) \propto \lambda^e(\sigma) \cdot \gamma^m(\sigma) = (\lambda \gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}$.

**Theorem:** When $\lambda \gamma > 6.83$ and $\gamma > 5.66$, there exists an $\alpha = \alpha(\lambda, \gamma)$ such that the particle system is $\alpha$-compressed at stationarity almost surely.

**Proof techniques.** Uses the cluster expansion and a Peierls argument.

**Theorem:** Moreover, separation occurs among the $\alpha$-compressed configurations at stationarity almost surely.

**Proof techniques.** Uses bridging [Miracle, Pascoe, Randall 2011] and a Peierls argument.
Results: Separation for large $\gamma$

**Theorem:** When $\lambda \gamma > 6.83$ and $\gamma > 5.66$, there exists an $\alpha = \alpha(\lambda, \gamma)$ such that the particle system is $\alpha$-compressed at stationarity almost surely.

Proof sketch.

Stationary distribution $\pi(\sigma) = (\lambda \gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}/Z$.

Let $S_\alpha$ be the non-$\alpha$-compressed configurations. Want to show $\pi(S_\alpha)$ is exponentially small.

Partition $S_\alpha$ into sets of configurations $A_k$ with the same perimeter $k$. Then:

$$\pi(A_k) = \sum_{\sigma \in A_k} (\lambda \gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}/Z$$

$$= (\lambda \gamma)^{-k} \cdot \sum_{\sigma \in A_k} \gamma^{-h(\sigma)}/Z$$
Results: Separation for large $\gamma$

**Theorem:** When $\lambda \gamma > 6.83$ and $\gamma > 5.66$, there exists an $\alpha = \alpha(\lambda, \gamma)$ such that the particle system is $\alpha$-compressed at stationarity almost surely.

Proof sketch (cont.)

So $\pi(A_k) = (\lambda \gamma)^{-k} \cdot \sum_{\sigma \in A_k} \gamma^{-h(\sigma)}/Z$.

If we had $\sum_{\sigma \in A_k} \gamma^{-h(\sigma)} \leq b^k$ for some $b > 1$, then:

$$\pi(S_\alpha) = \sum_{k=\alpha \cdot p_{\min}}^{p_{\max}} \pi(A_k) = \sum_{k=\alpha \cdot p_{\min}}^{p_{\max}} (\lambda \gamma)^{-k} \cdot \sum_{\sigma \in A_k} \gamma^{-h(\sigma)}/Z \leq \sum_{k=\alpha \cdot p_{\min}}^{p_{\max}} (\lambda \gamma)^{-k} \cdot b^k/Z$$

**Lemma [Volume-Surface Decomposition]:** When $\gamma > 5.66$, there are $a$ and $b$ such that:

$$a^n \cdot b^{-k} \leq \sum_{\sigma \in A_k} \gamma^{-h(\sigma)} \leq a^n \cdot b^k$$

However, while true in the uncolored case (compression), this is not true for our heterogeneous setting.
Results: Separation for large $\gamma$

**Lemma [Volume-Surface Decomposition]:** If $\Omega_\Lambda$ are all 2-colorings with monochrome perimeter of an uncolored configuration $\Lambda$ and $\gamma > 5.66$, then there are $a$ and $b$ such that:

$$a^n \cdot b^{-k} \leq \sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)} \leq a^n \cdot b^k$$

**Proof sketch (cont.)**

Express $\sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)}$ as a “polymer model.”

- An **interface $I$** between two color classes is a loop.
- Let $\Gamma_\Lambda$ be the set of all interfaces in $\Lambda$. Then:

$$\sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)} = \sum_{\text{pairwise disjoint } \Gamma' \in \Gamma_\Lambda} \prod_{I \in \Gamma'} \gamma^{-|I|}$$
Results: Separation for large $\gamma$

**Lemma [Volume-Surface Decomposition]:** If $\Omega_\Lambda$ are all 2-colorings with monochrome perimeter of an uncolored configuration $\Lambda$ and $\gamma > 5.66$, then there are $a$ and $b$ such that:

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Proof sketch (cont.)

$$\sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)} = \sum_{\text{pairwise disjoint } \Gamma' \in \Gamma_\Lambda} \prod_{I \in \Gamma'} \gamma^{-|I|}$$

A cluster is a multiset $X \subseteq \Gamma_\Lambda$ of connected interfaces. The cluster expansion for our quantity is:

$$\ln\left(\sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)}\right) = \sum_{X \subseteq \Gamma_\Lambda} \phi(X) \prod_{I \in X} \gamma^{-|I|}$$

Need to know this formal series converges. Also need to bound this sum.
Results: Separation for large $\gamma$

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**Proof sketch (cont.)**

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Using the Kotecký-Preiss condition with $\gamma > 5.66$ and a constant $c = 0.0001$, we show series convergence and:

$$a^n \cdot e^{-ck} \leq \sum_{\sigma \in \Omega_\Lambda} \gamma^{-h(\sigma)} \leq a^n \cdot e^{ck}$$
Results: Integration for $\gamma$ close to 1

Stationary distribution $\pi(\sigma) \propto \lambda^e(\sigma) \cdot \gamma^m(\sigma) = (\lambda \gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}$.

**Theorem:** When $\lambda(\gamma + 1) > 6.83$ and $0.98 \leq \gamma \leq 1.02$, there exists an $\alpha = \alpha(\lambda, \gamma)$ such that the particle system is $\alpha$-compressed at stationarity almost surely.

**Theorem:** Moreover, separation occurs among the $\alpha$-compressed configurations at stationarity with exponentially small probability.
Results: Integration for $\gamma$ close to 1

**Theorem:** When $\lambda(\gamma + 1) > 6.83$ and $0.98 \leq \gamma \leq 1.02$, there exists an $\alpha = \alpha(\lambda, \gamma)$ such that the particle system is $\alpha$-compressed at stationarity almost surely.

**Proof sketch.**

Recall: $\sum_{\sigma \in \Omega} \gamma^{-h(\sigma)} = \sum_{\text{pairwise disjoint } \Gamma' \in \Gamma} \prod_{I \in \Gamma'} \gamma^{-|I|}$

The $\gamma^{-|I|}$ term does not decay fast enough when $\gamma$ is close to 1.

Rewrite using the high temperature expansion.

$$\sum_{\sigma \in \Omega} \gamma^{-h(\sigma)} = \ldots \sum_{\text{even } E \subseteq E(\Lambda)} \left(\frac{\gamma-1}{\gamma+1}\right)^{|E|}$$

Then apply the cluster expansion and Peierls argument similar to the previous proof.
Open Questions

1. What is the mixing time of our algorithms?
   - Connections to the low temperature plus-boundary Ising model on $\mathbb{Z}^2$ suggests proofs are hard.
   - However, we observe compression in simulation after only $O(n^{3.3})$ iterations.

2. Are there critical values $\lambda^*$ and $\gamma^*$ marking phase transitions?

   ![Phase Transition Diagram]

   - No compression (expansion) $\lambda^* = 2 + \sqrt{2}$
   - Integration: $\gamma$ near 1
   - Separation: large $\gamma$

3. What other new ways can we use the cluster expansion?
   - Used to show aggregation/dispersion in the disconnected case. [Dutta, Li, Cannon, D., Aydin, Richa, Goldman, Randall]
Thank you!

sops.engineering.asu.edu
joshdaymude.wordpress.com