Elegant Derivatives of Large Products

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Introduction

This short writeup details two derivations of the solution:

$$\frac{d}{dx}\prod_{i=1}^{n} f_i(x) = \prod_{i=1}^{n} f_i(x) \cdot \sum_{i=1}^{n} \frac{f_i'(x)}{f_i(x)}$$

These derivations also work for infinite products:

$$\frac{d}{dx}\prod_{i=1}^{\infty} f_i(x) = \prod_{i=1}^{\infty} f_i(x) \cdot \sum_{i=1}^{\infty} \frac{f_i'(x)}{f_i(x)}$$

Method 1: Product Rule

The product rule states:

$$\frac{d}{dx}(u \cdot v) = u' \cdot v + u \cdot v'$$

By iteratively peeling off terms from the product and applying the product rule, we obtain:

$$\begin{aligned}
\frac{d}{dx}\prod_{i=1}^{n} f_i(x) &= \frac{d}{dx}\left(f_1(x) \cdot \prod_{i=2}^{n} f_i(x)\right) \\
&= f_1'(x) \cdot \prod_{i=2}^{n} f_i(x) + f_1(x) \cdot \frac{d}{dx}\left(\prod_{i=2}^{n} f_i(x)\right) \\
&= f_1'(x) \cdot \prod_{i=2}^{n} f_i(x) + f_1(x) \cdot \left(f_2'(x) \cdot \prod_{i=3}^{n} f_i(x) + f_2(x) \cdot \frac{d}{dx}\left(\prod_{i=3}^{n} f_i(x)\right)\right) \\
&= f_1'(x) \cdot \prod_{i=2}^{n} f_i(x) + f_1(x) \cdot \left(f_2'(x) \cdot \prod_{i=3}^{n} f_i(x) + f_2(x) \cdots\right. \\
&\quad \left. + f_{n-2}(x) \cdot \big(f_{n-1}'(x) \cdot f_n(x) + f_n'(x) \cdot f_{n-1}(x)\big) \cdots \right)
\end{aligned}$$

Distributing the singular $f_i(x)$ terms into their following nested sums and rearranging, we obtain

$$\frac{d}{dx}\prod_{i=1}^{n} f_i(x) = \sum_{i=1}^{n} f_i'(x) \cdot \frac{\prod_{j=1}^{n} f_j(x)}{f_i(x)} = \prod_{i=1}^{n} f_i(x) \cdot \sum_{i=1}^{n} \frac{f_i'(x)}{f_i(x)}.$$

Method 2: Leveraging Logarithms

Let $F(x) = \prod_{i=1}^{n} f_i(x)$. Then taking the natural logarithm of both sides yields

$$\ln(F(x)) = \ln\left(\prod_{i=1}^{n} f_i(x)\right) = \sum_{i=1}^{n} \ln(f_i(x))$$

The derivative of the natural logarithm is $\frac{d}{dx}\ln x = 1/x$. So, taking the derivative of both sides and observing that the derivative of a finite sum is equal to the finite sum of derivatives,

$$\begin{aligned} \frac{d}{dx}\ln(F(x)) &= \frac{d}{dx}\sum_{i=1}^{n}\ln(f_i(x)) \\ \frac{1}{F(x)}\cdot\frac{dF}{dx} &= \sum_{i=1}^{n}\frac{1}{f_i(x)}\cdot f_i'(x) \\ \frac{dF}{dx} &= F(x)\cdot\sum_{i=1}^{n}\frac{f_i'(x)}{f_i(x)} \end{aligned}$$

Substituting the full expression back in for $F(x)$ yields

$$\frac{d}{dx}\prod_{i=1}^{n}f_i(x) = \prod_{i=1}^{n}f_i(x)\cdot\sum_{i=1}^{n}\frac{f_i'(x)}{f_i(x)}$$

Notes and Caveats

- The second method only applies when $f_i(x) > 0$ for all i and x; otherwise, $\ln(f_i(x))$ is undefined. The first method holds in all cases.
- The second method relies on the derivative of a sum being equal to the sum of the derivatives. When applied to an infinite sum, this is true if (a) f_i is differentiable over the domain for all i, (b) $\sum_{i=1}^{\infty} f_i'(x)$ uniformly converges, and (c) there exists at least one point of convergence for $\sum_{i=1}^{\infty} f_i(x)$. Unless I'm mistaken, the first method does not have this problem and can be directly applied to infinite products.